

BCJ duality and the double copy in the soft limit

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Abstract

We examine the structure of infrared singularities in QCD and quantum General Relativity, from the point of view of the recently conjectured *double copy* property which relates scattering amplitudes in non-Abelian gauge theories with gravitational counterparts. We show that IR divergences in both theories are consistent with the double copy procedure, to all orders in perturbation theory, thus providing all loop-level evidence for the conjecture. We further comment on the relevance, or otherwise, to the so-called *dipole formula*, a conjecture for the complete structure of IR singularities in QCD.

1 Introduction

Scattering amplitudes in quantum field theories continue to generate a large amount of interest, due to the many phenomenological and theoretical applications. Much of this work (motivated by the relevance of QCD to hadron colliders) focuses on non-Abelian gauge theories, including supersymmetric counterparts (e.g. $\mathcal{N} = 4$ Super-Yang Mills theory) whose simpler structure provides a useful testing ground for theoretical techniques. Amplitudes have also been studied in gravitational field theories, namely General Relativity and its supersymmetric counterparts such as $\mathcal{N} = 8$ super-gravity, which may provide an ultraviolet finite field theory of gravity [1]. Although gravitational theories look superficially very different from non-Abelian gauge theories, a recent programme of work has suggested that gauge and gravity amplitudes may be related in an intriguing way [2–4]. Firstly, gauge theory amplitudes may be written in a special form which manifests an explicit duality between colour and kinematics, the so-called *BCJ duality* of [5]. Secondly, an m -point, L -loop order gauge theory amplitude in BCJ dual form can be translated into an equivalent gravity amplitude by the *double copy* procedure, in which colour factors are replaced by kinematic factors. That gauge theory amplitudes admit BCJ duality is known to be the case at tree level [3, 6–9], where the double copy property is equivalent to the known KLT relations [10] relating amplitudes in gauge and gravity theories. At loop level there is no formal proof of the double copy conjecture, although it is known that this should hold pending the existence of BCJ duality to all orders in the gauge theory, for the cases of pure gravity and $\mathcal{N} = 8$ super-gravity [3]. Loop-level checks of the duality have been carried out up to four loop order in $\mathcal{N} = 4$ SYM theory [2, 11–13] for amplitudes

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involving up to five external particles, and two loop order (for four point scattering with maximal helicity violation) in QCD [2].

As is perhaps clear from the above comments, much work on BCJ duality and the double copy property has focussed on $\mathcal{N} = 4$ SYM theory, in which a restricted set of loop diagrams contribute [4]. A significant motivation for studying $\mathcal{N} = 4$ SYM is that the double copy relates this to $\mathcal{N} = 8$ supergravity, so that one may investigate the issue of whether the latter theory is ultraviolet finite [1, 14, 15]. To date, less work has been invested in examining the non-supersymmetric context of pure Yang-Mills theory, where the double copy relates this to General Relativity. Examples include [16, 17] (see also [18]), in which amplitudes in $4 \leq \mathcal{N} \leq 8$ supergravity are considered by combining BCJ dual $\mathcal{N} = 4$ SYM amplitudes with those in a less supersymmetric theory, exploiting the fact that the two sets of kinematic factors in the double copy procedure need not both satisfy BCJ duality. Recently, an extension of the duality was considered, to Yang-Mills theory deformed by higher dimensional operators [19].

The aim of this paper is to point out that the infrared limit of QCD ² provides a clean environment in which to examine the double copy property in a non-supersymmetric context. We will argue that the known structure of infrared singularities in both QCD and gravity is consistent with the double copy, via BCJ duality, to all loop orders. The infrared behaviour of gauge theory amplitudes has been intensively studied over many years (see e.g. [20–29]). In QCD, the state of current knowledge regarding massless scattering amplitudes can be summarised in the so-called *dipole formula* [30–34], a conjecture which states that soft and collinear singularities exponentiate such that the exponent contains colour correlations between at most pairs of external legs. The formula is known to be exact up to two loops, and the structure of possible corrections at three loop order and beyond has been further studied in [35–39]. It is known the dipole formula fails already at two loop level for massive external particles [40–48], a fact which is not immediately relevant here due to the fact that the framework of BCJ duality and the double copy procedure is set up only for massless particles.

The study of infrared singularities in General Relativity was first examined in [49], and there has recently been a revival of interest, particularly in the work of [50–52] which aims to describe soft graviton physics in the same language used in a gauge theory context, and which we will review in what follows. It is now well established that the soft limit of gravity is *one loop exact*. That is, infrared singularities exponentiate such that the exponent contains only one loop graphs. Note that this implies that all IR singularities in gravity are dipoles, in the sense that at one loop in the exponent only pairs of particles can be correlated.

The above comments, and the relationship between gauge theory and gravity offered by the double copy procedure, beg the question: could the QCD dipole formula have an essentially gravitational origin? This thought motivated the present study, and we will see that in fact the answer appears to be no. Nevertheless, we will find that the structure of IR singularities in QCD match up at all orders with the known structure of singularities in General Relativity, providing all-loop-level evidence for the double copy conjecture. The argument is insensitive to possible corrections to the dipole formula - we will see explicitly that many singularities disappear when the double copy

²We consider pure gluodynamics (QCD without quarks) throughout the paper. For convenience, we refer to this as QCD throughout.

is taken. This includes collinear singularities, which are already known to vanish in gravity after summing over diagrams [49, 52].

The structure of the paper is as follows. In section 2 we review necessary background material in more detail, namely concepts relating to BCJ duality, the double copy, and the structure of infrared singularities in QCD and gravity. In section 3 we examine BCJ duality and the double copy in the soft limit at one loop order, before extending this analysis to two loop order in section 4. In section 5 we generalise our remarks to all loop orders, before summarising our results and concluding in section 6.

2 Review of necessary concepts

2.1 BCJ duality and the double copy

In this section we briefly summarise BCJ duality [5] and the associated double copy conjecture [2, 3], which together provide a map from gauge theory to gravity amplitudes, potentially at the multiloop level³.

A general massless m -point L -loop gauge theory amplitude $\mathcal{A}_m^{(L)}$ in D space-time dimensions can be written as

$$\mathcal{A}_m^{(L)} = i^L g^{m-2+2L} \sum_{i \in \Gamma} \int \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i C_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \quad (1)$$

where g is the coupling constant. Here the sum is over the complete set of graphs involving triple gluon vertices, consistent with the given loop order and number of external particles, and S_i a symmetry factor for each graph i (the dimension of its automorphism group). The denominator contains all relevant propagator momenta, and n_i is the kinematic numerator associated with each graph. Finally, C_i is the colour factor of each graph, obtained by dressing each three gluon vertex with a factor (adopting the same conventions as [4])

$$\tilde{f}^{abc} = i\sqrt{2}f^{abc}, \quad (2)$$

where f^{abc} are the usual SU(3) structure constants. Noteworthy points regarding this formula are:

- The restriction to graphs with cubic vertices only does not mean that graphs with four gluon vertices have been excluded. Rather, one can always choose to rewrite the latter in terms of cubic graphs, by introducing extra momenta in the denominator (which are compensated by additional factors in the numerator). This relies upon the fact that the colour factor for a four gluon vertex is a product of two structure constants, consistent with a pair of three gluon vertices.
- The numerators in eq. (1) may or may not come from individual Feynman diagrams. They may also have been obtained from e.g. a generalised unitary-based approach (see e.g. [53] and references therein), in which case each n_i collects all kinematic information associated with a particular scalar integral (fixed by the denominator).

³See also section II(A) of [4] for a recent and pedagogical review of this material.

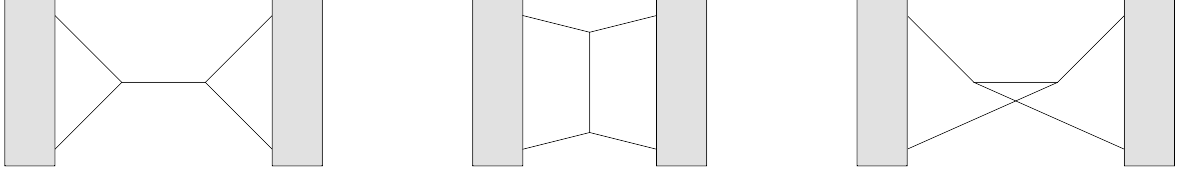


Figure 1: Illustration of three graphs related by BCJ transformations, where the grey boxes denote the rest of the amplitude.

From any graph at a given loop order, one may construct two other graphs related by taking *BCJ transformations* involving crossing of *s*-channel like subgraphs into *t* and *u*-channel subgraphs. This is best illustrated pictorially, as in figure 1. Such transformations exist for any internal line (propagator) in a given cubic graph, leading to a set of inter-related graphs, which can be generated from a single diagram. The colour factors associated with the three graphs in figure 1 satisfy the identity

$$c_s = c_t + c_u, \quad (3)$$

where c_i is the colour factor associated with the diagram containing an *i*-channel-like subgraph. This follows from the Jacobi identity for the structure constants, after factoring out the part of each colour factor which is the same for each graph. Note there is an ambiguity in how one defines signs in this identity (which can be compensated by a corresponding choice for the numerators). We here use the same conventions as [4].

It is conjectured that one can always choose to redefine the numerators n_i in the amplitude of eq. (1), such that they satisfy a similar identity to eq. (3). That is, the numerators of the *s*, *t* and *u*-channel-like graphs in figure 1 can be chosen to obey

$$n_s = n_t + n_u. \quad (4)$$

Furthermore, if a given colour factor picks up a minus sign under interchange of two (external or internal) legs, then

$$c_i \rightarrow -c_i \quad \Rightarrow \quad n_i \rightarrow -n_i. \quad (5)$$

This properties are collectively known as *BCJ duality*, after [5]. It has been proven at tree-level, but remains a conjecture at loop level. Transformations required to write the numerators in a suitable form are known as *generalised gauge transformations*, and an algorithmic procedure exists in principle to establish, for a given loop-level amplitude, what a possible set of numerators actually is [4]. It is not clear, however, whether this algorithm is fully general, particularly in non-supersymmetric theories where more diagrams enter than in supersymmetric cases. In general, one may represent the effect of a generalised gauge transformation on a given numerator n_i as

$$n_i \rightarrow n_i + \Delta_i, \quad (6)$$

where the $\{\Delta_i\}$ must satisfy

$$\sum_{i \in \Gamma} \frac{\Delta_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0. \quad (7)$$

Again the sum is over all cubic diagrams with colour factors c_i and propagator denominators $p_{\alpha_i}^2$. Indeed, eq. (7) is obtained by substituting the transformation of eq. (6) into eq. (1) and requiring that the amplitude is invariant. One may further decompose the generalised gauge parameters Δ_i (assuming these to be local) as [3]

$$\Delta_i = \sum_{\alpha_i} \Delta_{i,\alpha_i} p_{\alpha_i}^2. \quad (8)$$

That is, the Δ_i associated with diagram i can be expanded in terms of the inverse propagators of this diagram. The Δ_i factors then move contributions between diagrams by cancelling propagators, so that contributions from one diagram can be absorbed in another.

Once a form of the amplitude satisfying BCJ duality has been found, a remarkable conjecture states that one can turn it into a gravity amplitude, in a straightforward manner. By stripping off the colour factors in eq. (1) and replacing them with another set of numerators $\{\tilde{n}_i\}$, the *double copy conjecture* states that [2, 3]⁴

$$\mathcal{M}_m^{(L)} = i^{L+1} \left(\frac{\kappa}{\sqrt{2}} \right)^{m-2+2L} \sum_{i \in \Gamma} \int \prod_{l=1}^L \frac{d^D p_l}{(2\pi)^D} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{\prod_{\alpha_i} p_{\alpha_i}^2} \quad (9)$$

is an m -point, L -loop gravity amplitude. Note that the numerators $\{n_i\}$ and $\{\tilde{n}_i\}$ need not come from the same theory. In this paper, we will be concerned with both sets of numerators coming from QCD, in which case the gravity amplitude corresponds to general relativity (coupled to an antisymmetric tensor and a dilaton). If the two gauge theories are $\mathcal{N} = N$ and $\mathcal{N} = M$ Super-Yang-Mills theory, then one obtains an amplitude in $\mathcal{N} = (N + M)$ supergravity.

This is all we need for what follows. We now turn to the study of infrared singularities in QCD and gravity.

2.2 Infrared singularities in non-Abelian gauge theory

Infrared singularities in quantum field theory have been studied over many decades, with a vast accompanying literature. Here we briefly summarise only those facts which are of direct relevance to what follows. For a more detailed pedagogical review, see e.g. [34]. Whilst our statements can be interpreted in the context of a general non-Abelian gauge theory, we explicitly refer to QCD throughout.

It is by now well-known that infrared singularities factorise, such that the general structure of an m -point scattering amplitude in QCD is, schematically,

$$\mathcal{A}_m = \mathcal{H}_m \cdot \mathcal{S} \cdot \prod_{i=1}^m \frac{J_i}{\mathcal{J}_i}. \quad (10)$$

Here \mathcal{H}_m is an infrared finite *hard interaction*, which is dressed by a universal soft function \mathcal{S} that collects all infrared singularities. The *jet function* J_i collects collinear singularities associated with

⁴Note that in this paper we use $\kappa = \sqrt{16\pi G_N}$ rather than $\kappa = \sqrt{32\pi G_N}$ (see section 2.3). This modifies the coupling factors in eq. (9) relative to those in [2, 3].

external leg i , and the *eikonal jet function* \mathcal{J}_i removes the double counting of divergences which are both soft and collinear. The soft function is given by a vacuum expectation value of Wilson line operators along the space-time trajectories of the outgoing hard particles. Equivalently, one may calculate the soft part of a given hard interaction by dressing all external lines with all possible soft gluon emissions, according to the *eikonal Feynman rule*

$$g_s \mathbf{T}_i \frac{p^\mu}{p \cdot k}, \quad (11)$$

where g_s is the strong coupling constant, and p (k) the momentum of the hard external line (soft gluon) respectively. Furthermore, \mathbf{T}_i is a colour generator in the representation of external line i , where we have used the notation of [54, 55]. Where soft gluons meet off the external lines, they couple according to the usual three and four gluon vertices of QCD.

All singularities appearing in eq. (10) can be shown to exponentiate, so that instead of eq. (10) we may write

$$\mathcal{A}_m = \mathcal{H}_m \cdot Z, \quad (12)$$

where

$$Z = \exp \left[\sum_{n=1}^{\infty} c_n(\{p_i\}, \epsilon, \mu) \alpha_S^n \right] \quad (13)$$

contains all soft and collinear singularities, and depends upon the momenta $\{p_i\}$, as well as the dimensional regularisation parameter ϵ and scale μ . The structure of the exponent (i.e. the form of the coefficients $\{c_n\}$ in eq. (13)) is known explicitly up to two loop order for both massless and massive particles [40–48, 56, 57]. For massless particles, it has the remarkable property of involving both colour and kinematic correlations between at most pairs of particles, despite the fact that one would naïvely expect correlations between more than two particles to appear at two loop order and beyond. This property motivated the conjecture of the so-called *dipole formula* in QCD [30–34], which gives the all-order structure of eq. (13) as

$$Z = \exp \left\{ \int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \left[\frac{1}{8} \hat{\gamma}_K(\alpha_S(\lambda^2, \epsilon)) \sum_{(i,j)} \ln \left(\frac{2p_i \cdot p_j e^{i\pi\lambda_{ij}}}{\lambda^2} \right) \mathbf{T}_i \cdot \mathbf{T}_j - \frac{1}{2} \sum_{i=1}^m \gamma_{J_i}(\alpha_S(\lambda^2, \epsilon)) \right] \right\}. \quad (14)$$

Here $\hat{\gamma}_K$ is the cusp anomalous dimension (itself a perturbative expansion in α_S with constant coefficients), and γ_{J_i} a further anomalous dimension associated with jet i , and which collects hard collinear contributions. The double sum in the first term is over all pairs of particles (i, j) , and following [34] we have used the notation $\lambda_{ij} = 1$ if i and j are both in the initial or both in the final state (otherwise $\lambda_{ij} = 0$). Equation (14) indeed contains correlations between dipoles only, and is known to break down already at two loop order for massive external legs. For the massless case, corrections may occur at three loop order and beyond, either through explicit kinematic dependence of the relevant Feynman integrals, or through higher order Casimir invariants appearing in the cusp anomalous dimension. The form of possible corrections has been investigated in [35–39]. Further progress may be made using recently developed techniques for classifying the structure of the exponent [28, 29, 58, 59], or by considering alternative gauges [60].

2.3 Infrared singularities in gravity

Complementary to the above mentioned studies in gauge theory, IR singularities have also been investigated in gravity, commencing with the classic work of [49]. Recently, there has been a revival of interest, which has focused in particular on writing the structure of gravitational IR divergences using the same language as is used in modern QCD studies. Reference [50] suggested the use of the following gravitational generalisation of eq. (10):

$$\mathcal{M}_m = \mathcal{H}_m \cdot \mathcal{S}^{\text{grav}}, \quad (15)$$

where \mathcal{M}_m is an m -point gravity amplitude. Here \mathcal{H}_m is again a hard interaction which is infrared finite, and $\mathcal{S}^{\text{grav}}$ is a universal gravitational soft function, which is given by a vacuum expectation value of suitable Wilson line operators. There are no jet functions, due to the fact that collinear singularities cancel in gravity after summing over all diagrams and using momentum conservation. The latter property was first established in the soft limit [49], but has recently been fully extended to encompass hard collinear singularities [52, 61]. Furthermore, ref. [51] examined the form of eq. (15) in more detail using the path integral approach of [62], also classifying what happens beyond the eikonal approximation. A similar structure of next-to-eikonal corrections was found as in the case of gauge theory, as explored in [62, 63]. Gravitational Wilson lines in a soft-graviton context were further explored in [64] using the radial coordinate space picture of [60], and a continuous deformation was found between the cusp anomalous dimensions of QED and gravity at one loop.

As in the gauge theory case, the soft function in gravity exponentiates. However, a drastic simplification over gauge theory occurs in that the exponent contains only one-loop diagrams, with no higher order corrections. This property is known as *one-loop exactness*, and has been firmly established by the studies of [49, 50, 52]. It implies that all infrared singularities (i.e. to all orders in perturbation theory) ultimately stem from the exponentiation of the one-loop corrections, in marked contrast to the gauge theory case of eq. (13): even if the dipole formula of eq. (14) happens to be true, there is still a further perturbation expansion to be carried out in the exponent, whose soft part requires the cusp anomalous dimension to all orders. It is amusing to note, as has already been mentioned in the introduction, that one-loop exactness in gravity implies that all infrared singularities are associated with pairs of particles (the most that can be correlated with a single graviton exchange). This is reminiscent of the QCD dipole formula in some sense (i.e. that higher multipole correlations vanish), and allows us to speculate as to whether the dipole formula may have a gravitational origin. The results of this paper would appear to suggest that this is not the case, due to the disappearance of many singularities upon taking the double copy of a gauge theory. However, one-loop exactness has another important role: it tells us that we know the all-order structure of IR singularities in gravity completely. We can then ask whether the known IR divergence structures in QCD and gravity are consistent with each other, if we apply the double copy procedure (via BCJ duality). If this is so, this provides all-order evidence (at least in a particular limit) for the double copy conjecture. This will be the aim of the rest of the paper.

In our subsequent calculations we will use the following conventions (a number of different choices exist in the literature - see e.g. [65] for a convenient reference). The graviton field $h_{\mu\nu}$ is defined in terms of the metric tensor $g_{\mu\nu}$ via

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \quad (16)$$

where $\eta_{\mu\nu}$ is the Minkowski space metric, $\kappa = \sqrt{16\pi G_N}$ and G_N is Newton's constant. Emission of soft gravitons with momentum k from a hard external line of momentum p is then given by the eikonal Feynman rule (see [64] for a recent derivation using the above conventions)

$$\frac{\kappa}{2} \frac{p_\mu p_\nu}{p \cdot k}, \quad (17)$$

which may be compared with the gauge theory eikonal Feynman rule of eq. (11). Finally, we will need the graviton propagator, for which we use the result in the de Donder gauge:

$$D_{\mu\nu,\alpha\beta}(k) = \frac{-iP_{\mu\nu,\alpha\beta}}{k^2 - i\epsilon}, \quad P_{\mu\nu,\alpha\beta} = \eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \frac{2}{D-2}\eta_{\mu\nu}\eta_{\alpha\beta}. \quad (18)$$

Having reviewed the necessary material for what follows, we begin our investigation of the double copy procedure in the soft limit in the following section.

3 One loop analysis

In the previous section, we reviewed various theoretical ideas concerning BCJ duality, the double copy, and the structure of infrared singularities in both QCD and gravity. This motivated the central question of our study: do the known IR singularity structures in QCD and gravity match up with each other upon taking the double copy of QCD? The aim of this paper is to argue that this is indeed the case, and we will begin at one-loop order with the following strategy:

1. We classify all possible BCJ relations of the form of eq. (4), between sets of three diagrams containing cubic vertices, in the case of pure Yang-Mills theory.
2. Next, we write down all possible diagrams which give infrared singularities at this order, and show that they can be matched up in sets of three, each of which offers an explicit solution of a BCJ relation in the soft limit.
3. Having shown that BCJ duality is satisfied (up to corrections which are subleading in the soft limit), we then take the double copy of all infrared divergent diagrams, and verify that this reproduces the one-loop divergences in GR.

The first step is to obtain the BCJ relations, which we do in the following subsection.

3.1 BCJ relations

To start with, we must write down all possible one-loop graphs, and for a concrete example we consider the case of 4-point scattering, labelled as in figure 2. We may also define the usual Mandelstam invariants

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2. \quad (19)$$

The box and triangle graphs are labelled in figure 3. The bubble graphs are shown in figure 4.

As explained in section 2, the diagrams in figures 3 and 4 form sets of three, related by subjecting internal lines to a BCJ transformation (replacement of t -channel-like exchange by s - and u -channel

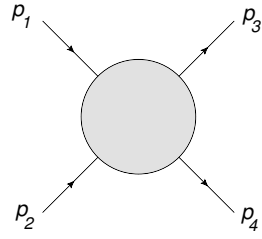


Figure 2: Four point scattering, where all momenta are taken to be outgoing.

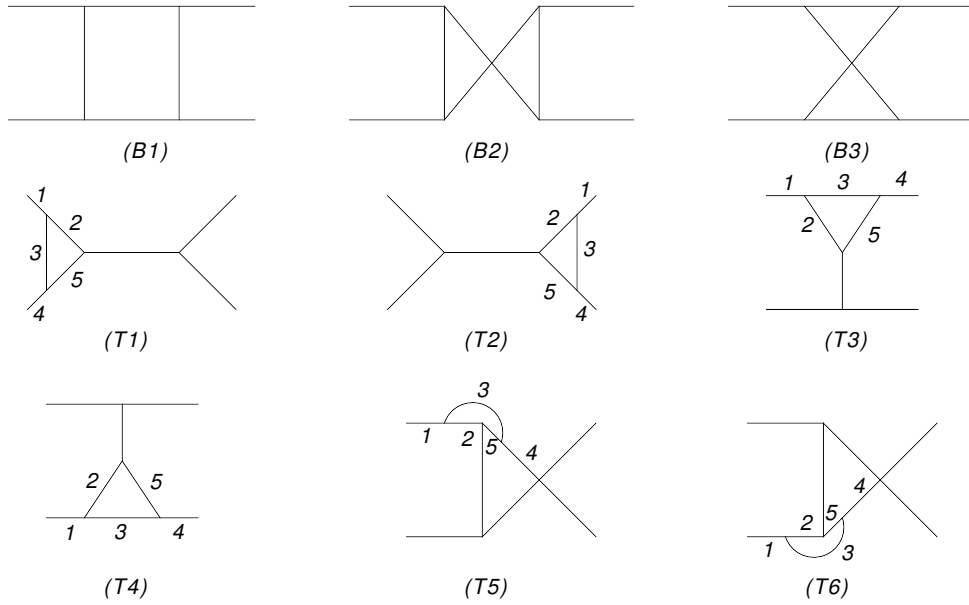


Figure 3: Set of all box and triangle graphs at one-loop order, where the relevance of the numeric labels is explained in the text.

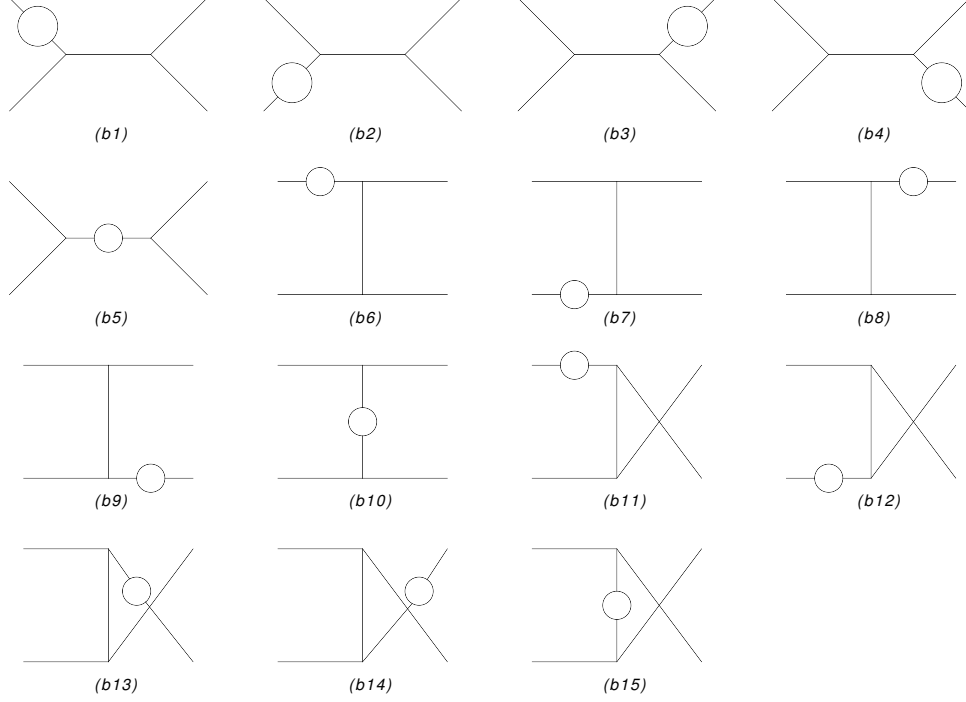


Figure 4: Set of bubble graphs at one-loop order.

exchanges). The colour factors of such a set satisfy the Jacobi identity, and BCJ duality, if satisfied, then requires a corresponding equation to hold for the kinematic numerators associated with each graph. The BCJ relations in the present case can be split into two kinds. Firstly, there are relations which relate two box topologies to a triangle. These are (using the labels from figures 3 and 4)

$$\begin{aligned}
-n_{B1} + n_{B2} + n_{T1} &= 0, \\
-n_{B1} + n_{B2} + n_{T3} &= 0, \\
-n_{B1} + n_{B2} + n_{T5} &= 0, \\
-n_{B1} + n_{B2} + n_{T7} &= 0, \\
-n_{B3} + n_{B2} + n_{T9} &= 0, \\
-n_{B3} + n_{B2} + n_{T11} &= 0.
\end{aligned} \tag{20}$$

Next, there are BCJ relations that relate two triangle topologies with a bubble. There are two subclasses - firstly those relations which involve an external self-energy:

$$\begin{aligned}
n_{T1}(1, 2, 3, 4, 5) + n_{T1}(1, 3, 2, 4, 5) - n_{b1} &= 0, \\
n_{T1}(1, 2, 3, 4, 5) + n_{T1}(1, 2, 5, 4, 3) - n_{b2} &= 0, \\
n_{T2}(1, 2, 3, 4, 5) + n_{T2}(1, 3, 2, 4, 5) - n_{b3} &= 0, \\
n_{T2}(1, 2, 3, 4, 5) + n_{T2}(1, 2, 5, 4, 3) - n_{b4} &= 0, \\
n_{T3}(1, 2, 3, 4, 5) + n_{T3}(1, 3, 2, 4, 5) - n_{b6} &= 0, \\
n_{T3}(1, 2, 3, 4, 5) + n_{T3}(1, 2, 5, 4, 3) - n_{b8} &= 0,
\end{aligned}$$

$$\begin{aligned}
n_{T4}(1, 2, 3, 4, 5) + n_{T4}(1, 3, 2, 4, 5) - n_{b7} &= 0, \\
n_{T4}(1, 2, 3, 4, 5) + n_{T4}(1, 2, 5, 4, 3) - n_{b9} &= 0, \\
n_{T5}(1, 2, 3, 4, 5) + n_{T5}(1, 3, 2, 4, 5) - n_{b11} &= 0, \\
n_{T5}(1, 2, 3, 4, 5) + n_{T5}(1, 2, 5, 4, 3) - n_{b13} &= 0, \\
n_{T6}(1, 2, 3, 4, 5) + n_{T6}(1, 3, 2, 4, 5) - n_{b12} &= 0, \\
n_{T6}(1, 2, 3, 4, 5) + n_{T6}(1, 2, 5, 4, 3) - n_{b14} &= 0.
\end{aligned} \tag{21}$$

Here we have been careful to label the momenta associated with each leg in the triangles, as shown in figure 3. Effecting a BCJ transformation on internal lines in the triangle graphs can “twist” the triangle. The resulting numerator factor may be written in terms of the numerator of the original triangle topology, but with leg labels interchanged. For brevity, we do not bother labelling momentum legs for the bubble and box graphs.

Secondly, there are those which involve an internal self-energy:

$$\begin{aligned}
n_{T1}(1, 2, 3, 4, 5) + n_{T1}(1, 5, 3, 4, 2) - n_{b5} &= 0, \\
n_{T2}(1, 2, 3, 4, 5) + n_{T2}(1, 5, 3, 4, 2) - n_{b5} &= 0, \\
n_{T3}(1, 2, 3, 4, 5) + n_{T3}(1, 5, 3, 4, 2) - n_{b10} &= 0, \\
n_{T4}(1, 2, 3, 4, 5) + n_{T4}(1, 5, 3, 4, 2) - n_{b10} &= 0, \\
n_{T5}(1, 2, 3, 4, 5) + n_{T5}(1, 5, 3, 4, 2) - n_{b15} &= 0, \\
n_{T6}(1, 2, 3, 4, 5) + n_{T6}(1, 5, 3, 4, 2) - n_{b15} &= 0.
\end{aligned} \tag{22}$$

Again we have been careful to label the momenta where necessary. In fact, the colour factors of a given triangle pick up a minus sign (due to antisymmetry of the structure constants) under interchange of any of the three legs (2,3,5), as labelled in figure 3. BCJ duality (from eq. (5)) then dictates that the corresponding numerators should obey similar relations. This implies that the first two terms of each relation in eqs. (21) and (22) cancel each other, leaving

$$n_{bi} = 0 \quad \forall \quad i. \tag{23}$$

That is, the numerators for all bubble graphs should be zero. The numerators for the triangles are then fixed by the relations of eq. (20), each involving two boxes and a triangle.

Having obtained the BCJ relations, one can show that these are automatically satisfied in the soft limit, in a Feynman gauge calculation. This is the subject of the following subsection.

3.2 Soft limit

In this section, we consider all possible infrared singular diagrams at one loop order, and show that their numerators explicitly satisfy the BCJ relations obtained in the previous section. IR singularities at one loop are generated by dressing tree level diagrams with soft gluon emissions. Considering once again the concrete example of 4-point scattering, we may write the full tree level

amplitude as⁵

$$\mathcal{A}^{(0)} = \sum_{x \in \{s, t, u\}} \mathcal{A}_x, \quad \mathcal{A}_x = \frac{c_x n_x}{x} \quad (24)$$

where $\{n_x\}$ is a suitable set of numerators (with colour factors $\{c_i\}$) obeying the tree-level BCJ relations

$$n_s + n_u - n_t = 0, \quad c_s + c_u - c_t = 0. \quad (25)$$

Note that we have absorbed coupling factors into the tree level numerators, so that all powers of the coupling in subsequent equations correspond to higher order corrections. Also, implicit in eq. (24) is the fact that the four-gluon vertex graph has been rewritten in terms of cubic graphs. The effect of dressing a tree-level amplitude \mathcal{A}_x with a soft gluon exchanged between legs i and j is given by

$$g_s^2 \mathbf{T}_i \cdot \mathbf{T}_j \mathcal{I}_{ij} \mathcal{A}_x,$$

where the eikonal integral factor

$$\mathcal{I}_{ij} = i \int \frac{d^D k}{(2\pi)^D} \frac{p_i \cdot p_j}{k^2 p_i \cdot k p_j \cdot k}, \quad (26)$$

as results from connecting two eikonal Feynman rules (eq. (11)) with a gluon propagator (note we use the Feynman gauge). The full infrared singular part of the one-loop amplitude can then be written⁶

$$\mathcal{A}_{\text{IR}}^{(1)} = i g_s^2 \sum_x \sum_{i < j} \mathbf{T}_i \cdot \mathbf{T}_j \mathcal{I}_{ij} \frac{\mathcal{A}_x}{x}. \quad (27)$$

Here the first sum is over the three tree-level cubic topologies, and the second sum is over all pairs of external legs, where each pair is counted only once. There are no contributions from soft emissions which begin and end on the same external line, due to the fact that all outgoing hard particles are massless ($p_i^2 = 0$). Furthermore, there are no contributions from internal self-energy diagrams: such graphs contain additional hard propagators that remove the infrared singularity.

We can now interpret eq. (27) from the viewpoint of BCJ duality. Firstly, the lack of internal and external self-energy graphs means that the numerators for these graphs may be set to zero in the soft limit. We may write this as

$$\hat{n}_{bi} = 0 \quad \forall \quad i, \quad (28)$$

where the hat above the numerator implies that we are evaluating this only in the soft limit. Such numerators are defined only up to arbitrary terms containing at least one power of a soft gluon momentum k , as this kills off the infrared singularity generated by the integration over k . Equation (28) indeed satisfies the requirement of eq. (23).

⁵Note that we here adopt the phase conventions of eq. (1). Associating a factor i and $-i$ with each propagator and vertex respectively, one may absorb a further factor of $1/i$ in the numerators n_x to obtain an overall power of i^L at L -loop order. Performing the double copy involves an extra numerator, and thus an additional explicit factor of i in the gravity amplitude.

⁶Strictly speaking the eikonal integral of eq. (26) is zero, being a scaleless integral in dimensional regularisation. However, this is due to the cancellation of the IR divergence with a spurious UV pole which results from the replacement of quadratic propagators by linear ones, such that one may recover the IR divergence using an additional regularisation procedure.

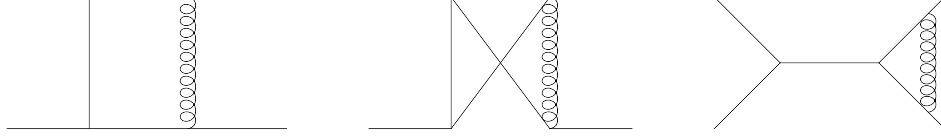


Figure 5: Diagrams involving a soft gluon exchange between legs 3 and 4, where solid (curly) lines denote hard (soft) gluons respectively.

Secondly, the terms appearing in eq. (27) can be grouped into sets of three, whose numerators satisfy the remaining BCJ relations. Interchanging the orders of summation in eq. (27), one may pick out a particular pair (i, j) , to give

$$\sum_x g_s^2 \mathbf{T}_i \cdot \mathbf{T}_j \int \frac{d^D k}{(2\pi)^D} \frac{p_i \cdot p_j}{k^2 p_i \cdot k p_j \cdot k} \frac{c_x n_x}{x}, \quad (29)$$

where we have substituted in the explicit forms for \mathcal{I}_{ij} and \mathcal{A}_x from eqs. (24) and (26). This is three separate terms, each of which can be interpreted as a soft limit of one of the graphs appearing in figure 3. Taking the pair (3,4) as an example, we may interpret eq. (29) as shown in figure 5. From eq. (29), we may associate with each term a collected numerator and colour factor

$$\hat{n}_{x,ij} = (p_i \cdot p_j) n_x, \quad c_{x,ij} = \mathbf{T}_i \cdot \mathbf{T}_j c_x. \quad (30)$$

One may then form the relations

$$\hat{n}_{s,ij} + \hat{n}_{u,ij} - \hat{n}_{t,ij} = 0, \quad c_{s,ij} + c_{u,ij} - c_{t,ij} = 0, \quad (31)$$

which follow from the fact that the tree level colour factors and numerators satisfy eq. (25), and that these are multiplied by a common factor in eq. (30). Equation (31) contains six different relations (the number of ways of choosing two external legs out of four), and correspond precisely to soft limits of the BCJ relations in eq. (20). Take, for example, the case shown in figure (5). These diagrams consist of soft limits of graphs $(B1)$, $(B2)$ and $(T3)$ in figure 3, and the relevant BCJ relation from eq. (31) thus corresponds to the second relation in eq. (20).

We have now seen that all of the BCJ relations derived in the previous section (for the case of full QCD) are satisfied in the soft limit. For strict BCJ duality to hold, however, we must also demonstrate that the numerators are consistent with eq. (5), namely that appropriate sign changes occur under interchange of internal legs. That this is indeed the case is described in appendix A.

Armed with the above knowledge, we are now permitted to take the double copy of eq. (27) as prescribed in [2], to give ⁷

$$\mathcal{M}_{\text{IR}}^{(1)} = - \sum_x \sum_{i < j} \left(\frac{\kappa}{\sqrt{2}} \right)^2 \int \frac{d^D k}{(2\pi)^D} \frac{(p_i \cdot p_j)^2}{k^2 p_i \cdot k p_j \cdot k} \frac{n_x n_x}{x}$$

⁷Here we have again adopted the phase conventions of eq. (1) - see the footnote on p. 12.

$$= - \sum_x \sum_{i < j} \left(\frac{\kappa}{2}\right)^2 \int \frac{d^D k}{(2\pi)^D} \frac{2(p_i \cdot p_j)^2}{k^2 p_i \cdot k p_j \cdot k} \frac{n_x n_x}{x}. \quad (32)$$

If the double copy procedure is valid, this result should give the infrared singular parts of the 4-point, 1 loop amplitude in GR. That this is indeed the case can be seen as follows. Firstly, the factor

$$\frac{in_x n_x}{x}$$

is, by the tree-level double copy procedure, a gravitational tree level graph for the s , t or u topology⁸. This is then dressed by an eikonal integral, which corresponds to the appropriate gravitational generalisation of eq. (26). To check this, one may contract eikonal Feynman rules on legs i and j with the propagator of eq. (18) to get

$$\begin{aligned} \mathcal{I}_{ij}^{\text{grav.}} &= \left(\frac{\kappa}{2}\right)^2 \int \frac{d^D k}{(2\pi)^D} \frac{p_i^\mu p_i^\nu}{p_i \cdot k} \left(-\frac{p_j^\mu p_j^\nu}{p_j \cdot k}\right) \left(-\frac{i}{k^2}\right) \left(\eta_{\mu\alpha} \eta_{\nu\beta} + \eta_{\mu\beta} \eta_{\nu\alpha} - \frac{2}{D-2} \eta_{\mu\nu} \eta_{\alpha\beta}\right) \\ &= i \left(\frac{\kappa}{2}\right)^2 \int \frac{d^D k}{(2\pi)^D} \frac{2(p_i \cdot p_j)^2}{k^2 p_i \cdot k p_j \cdot k}, \end{aligned} \quad (33)$$

where the minus sign in the second eikonal Feynman rule results from reversing the sign of the soft gluon momentum. Dressing the tree-level gravitational interaction with equation (33) is in exact agreement with eq. (32).

Note that, as first observed in [49], soft collinear singularities cancel after summing over all diagrams. Taking those which are associated with line i as an example, these are generated by the sum of eikonal integrals

$$\sum_{j \neq i} \int \frac{d^D k}{(2\pi)^D} \lim_{k \rightarrow p_i} \left[\frac{4(p_i \cdot p_j)^2}{k^2 p_i \cdot k p_j \cdot k} \right] = \sum_{j \neq i} \int \frac{d^D k}{(2\pi)^D} \lim_{k \rightarrow p_i} \left[\frac{4(p_i \cdot p_j)}{k^2 p_i \cdot k} \right]. \quad (34)$$

Applying momentum conservation

$$p_i = - \sum_{j \neq i} p_j \quad (35)$$

then gives zero on the right-hand side of eq. (34), up to non-singular terms. Note that the QCD equivalent of eq. (34) (also including colour factors) is

$$\sum_{j \neq i} \mathbf{T}_i \cdot \mathbf{T}_j \int \frac{d^D k}{(2\pi)^D} \lim_{k \rightarrow p_i} \left[\frac{(p_i \cdot p_j)}{k^2 p_i \cdot k p_j \cdot k} \right] = \sum_{j \neq i} \mathbf{T}_i \cdot \mathbf{T}_j \int \frac{d^D k}{(2\pi)^D} \lim_{k \rightarrow p_i} \left[\frac{1}{k^2 p_i \cdot k} \right]. \quad (36)$$

Colour conservation

$$\mathbf{T}_i = - \sum_{j \neq i} \mathbf{T}_j \quad (37)$$

then gives

$$\sum_{j \neq i} \mathbf{T}_i \cdot \mathbf{T}_j \int \frac{d^D k}{(2\pi)^D} \lim_{k \rightarrow p_i} \left[\frac{(p_i \cdot p_j)}{k^2 p_i \cdot k p_j \cdot k} \right] = C_i \int \frac{d^D k}{(2\pi)^D} \lim_{k \rightarrow p_i} \left[\frac{1}{k^2 p_i \cdot k} \right], \quad (38)$$

⁸Recall that we absorbed gauge theory coupling factors into the tree-level numerators. Implicit in eq. (32) is that these have been replaced appropriately in the double copy to gravity.

where C_i is the quadratic Casimir associated with external line i ($C_i = C_A = N_c$ for pure gluodynamics). This remains singular, and a simple physical way to interpret this is that collinear singularities can only depend on the quantum numbers of a single particle, which include the squared charge of the appropriate external line. In QCD, this is the quadratic Casimir operator in the relevant representation, whereas in gravity this is the squared 4-momentum, which vanishes if collinear singularities are to be present (i.e. if the leg is massless). The double copy procedure has here provided an interesting mechanism for the cancellation of soft-collinear singularities on the gravity side: the colour dependence in QCD gets replaced by additional momentum factors, which generate the necessary squared 4-momentum. Note that this means that there are singularities on the gauge theory side that vanish upon performing the double copy. We will see this happening more generally at two-loop order and beyond.

We have now verified that the soft limit of GR is precisely reproduced upon taking the double copy of the soft limit of QCD at one-loop order. Although we here focused on the particular case of 4-point scattering, the argument is easily generalised to any number of external legs. Having examined the one-loop case, we proceed to two loops in the following section.

4 Two loop analysis

In the previous section, we showed that the known infrared singularities of GR are reproduced by double copying those of QCD at one-loop order. Application of the double copy relied upon the fact that the BCJ relations could be satisfied in the soft limit. In fact, solution of these relations at this order was rather straightforward, and relied ultimately on the fact that they are satisfied at tree level. Put another way, the soft BCJ relations of eq. (31) all had the form of a common eikonal factor $(p_i \cdot p_j)$ multiplying a tree-level relation. This simple structure will no longer be the case at two loop order. We will see that numerators for individual graphs, computed in the Feynman gauge, do not automatically satisfy the BCJ relations in the soft limit. However, this will turn out to be irrelevant for reproducing the known infrared singularities of GR.

Our strategy will be the same as at one-loop order, and begins by writing down the BCJ relations in full QCD. Then one draws all possible soft topologies, and demonstrates that the resulting numerators satisfy the BCJ relations in the soft limit. Our experience at one loop tells us that we do not have to worry about most of the bubble diagrams. The numerators for these vanish as before, which will again be satisfied in the soft limit by the fact that internal and external self-energies are IR-suppressed. The only exception to this is the presence of self-energies associated with soft gluons, as we will see in what follows. There is a large number of diagrams at two-loop order and we do not collect them all here. Rather, we consider directly relevant soft topologies, and show that these can indeed be matched up with BCJ relations.

We consider tree-level m -point scattering dressed by soft gluons up to two-loop order. Possible soft topologies can then link two, three or four external lines. Examples are shown in figure 6, where we label each topology. We may also write the eikonal integrals that go with each soft topology, as

$$\mathcal{I}_{ijkl}^{(||_4)} = - \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{p_i \cdot p_j p_k \cdot p_l}{k^2 l^2 p_i \cdot k p_j \cdot k p_k \cdot l p_l \cdot l},$$

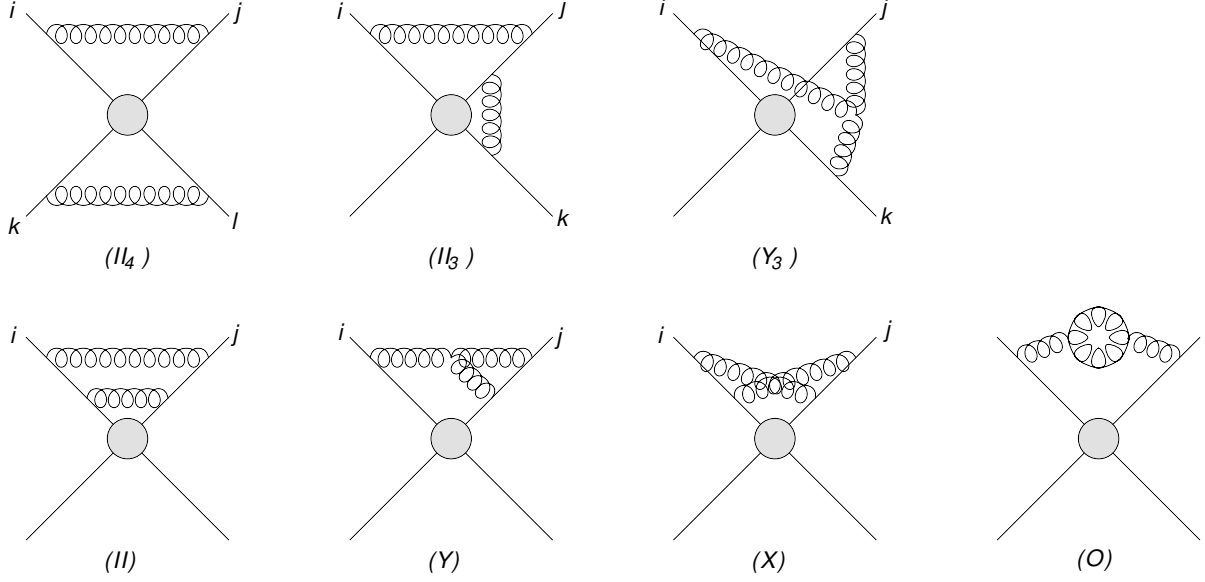


Figure 6: Two-loop soft topologies in m -point scattering, where only four external lines are shown.

$$\begin{aligned}
\mathcal{I}_{ijk}^{(ll_3)} &= - \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{p_i \cdot p_j p_j \cdot p_k}{k^2 l^2 p_i \cdot k p_j \cdot k p_j \cdot (k+l) p_k \cdot l}, \\
\mathcal{I}_{ijk}^{(Y_3)} &= - \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{V_{\mu\nu\rho}(k, l) p_i^\mu p_j^\nu p_k^\rho}{k^2 l^2 (k+l)^2 p_i \cdot k p_j \cdot l p_k \cdot (k+l)}, \\
\mathcal{I}_{ij}^{(ll)} &= - \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{(p_i \cdot p_j)^2}{k^2 l^2 p_i \cdot k p_i \cdot (k+l) p_j \cdot k p_k \cdot (k+l)}, \\
\mathcal{I}_{ij}^{(Y)} &= - \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{V_{\mu\nu\rho}(k, l) p_i^\mu p_j^\nu p_j^\rho}{k^2 l^2 (k+l)^2 p_i \cdot k p_j \cdot l p_j \cdot k}, \\
\mathcal{I}_{ij}^{(X)} &= - \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{(p_i \cdot p_j)^2}{k^2 l^2 p_i \cdot k p_i \cdot (k+l) p_j \cdot l p_k \cdot (k+l)}, \\
\mathcal{I}_{ij}^{(O)} &= - \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{V_{\mu\nu\rho}(k, l) V_{\nu\sigma\rho}(k, l) p_i^\mu p_j^\sigma}{(k^2)^2 l^2 (l-k)^2 p_i \cdot k p_j \cdot k},
\end{aligned} \tag{39}$$

where

$$V_{\mu\nu\rho}(k, l) \sim \mathcal{O}(k, l) \tag{40}$$

is the three-gluon vertex, which in this case couples together only soft momenta. Our notation for the eikonal integrals specifies which indices are coupled together by soft gluon emissions, and note that the ordering of these indices can be important (e.g. $\mathcal{I}_{ij}^{(Y)}$ is not the same as $\mathcal{I}_{ji}^{(Y)}$: the latter corresponds to a Y graph which is reflected with respect to the former). The infrared singular part of the two-loop gauge theory amplitude can now be written as

$$\mathcal{A}_{\text{IR}}^{(2)} = -g_s^4 \sum_{x \in \{s, t, u\}} \left[\sum_{\langle ijkl \rangle} \mathbf{T}_i \cdot \mathbf{T}_j \mathbf{T}_k \cdot \mathbf{T}_l \mathcal{I}_{ijkl}^{(ll_4)} + \sum_{\langle ijk \rangle} \left(\mathbf{T}_i \cdot \mathbf{T}_j \mathbf{T}_j \cdot \mathbf{T}_k \mathcal{I}_{ijk}^{(ll_3)} + \tilde{f}^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_k^c \mathcal{I}_{ijk}^{(Y_3)} \right) \right]$$

$$\begin{aligned}
& + \sum_{\langle ij \rangle} \left((\mathbf{T}_i \cdot \mathbf{T}_j)^2 \mathcal{I}_{ij}^{(||)} + \tilde{f}^{abc} \mathbf{T}_i^a \mathbf{T}_j^b \mathbf{T}_j^c \mathcal{I}_{ij}^{(Y)} + \mathbf{T}_i^a \mathbf{T}_i^b \mathbf{T}_j^a \mathbf{T}_j^b \mathcal{I}_{ij}^{(X)} + \tilde{f}^{abc} \tilde{f}^{bdc} \mathbf{T}_i^a \mathbf{T}_j^d \mathcal{I}_{ij}^{(O)} \right) \Bigg] \\
& \times \frac{n_x}{x},
\end{aligned} \tag{41}$$

where the notation $\langle ij \dots k \rangle$ denotes that one must sum over all distinct multiples, and we have used the colour vertex factor of eq. (2). This expression is obtained by dressing the tree-level interaction with all possible eikonal integrals, and is the two-loop generalisation of eq. (27). Furthermore, we have taken a factor of $i^2 = -1$ out of the eikonal integrals, so as to make manifest the phase convention of eq. (1).

As in the one-loop case, we can match up terms in eq. (41) into sets of three, such that they correspond to the soft limit of a BCJ relation. There are two distinct classes of relation. Firstly, there are relations in which the same eikonal integral dresses different tree-level interaction graphs. This is the only scenario that was possible at one-loop order, and the relevant three graphs are obtained by picking a particular term in eq. (41), and keeping the sum over tree-level topologies x . As an example, consider the contribution

$$g_s^4 \sum_x (\mathbf{T}_i \cdot \mathbf{T}_j)^2 \mathcal{I}_{ij}^{(||)}, \tag{42}$$

obtained by selecting a particular pair in the first term in the second line of eq. (41). This corresponds to the graphs shown in figure 7(a). These indeed correspond to the soft limit of three full QCD graphs which enter a BCJ relation, namely those shown in figure 7(b). From eq. (42), one may associate a collected numerator and colour factor for each graph, according to

$$c_x^{(||)} = (\mathbf{T}_i \cdot \mathbf{T}_j)^2 c_x, \quad \hat{n}_x^{(||)} = (p_i \cdot p_j)^2 n_x, \tag{43}$$

where the hat once again reminds us that such numerators are defined in the soft limit, and may have arbitrary terms $\sim \mathcal{O}(k, l)$ added. The numerators thus defined satisfy the relations

$$\hat{n}_s^{(||)} + \hat{n}_u^{(||)} - \hat{n}_t^{(||)} = 0, \tag{44}$$

by virtue of the fact that this is satisfied by the tree-level numerators, and that these are all multiplied by a common factor.

It is clear that the above argument generalises to any set of three graphs which have the same soft topology Z . That is, one may form soft numerators

$$\hat{n}_x^{(Z)} = \mathcal{I}_{ij\dots k}^{(Z)} \Big|_{\text{num.}} n_x, \tag{45}$$

where $\mathcal{I}_{ij\dots k}^{(Z)} \Big|_{\text{num.}}$ denotes the numerator of the relevant eikonal integral, such that

$$\hat{n}_s^{(Z)} + \hat{n}_u^{(Z)} - \hat{n}_t^{(Z)} = 0. \tag{46}$$

This then corresponds to the soft limit of a BCJ relation.

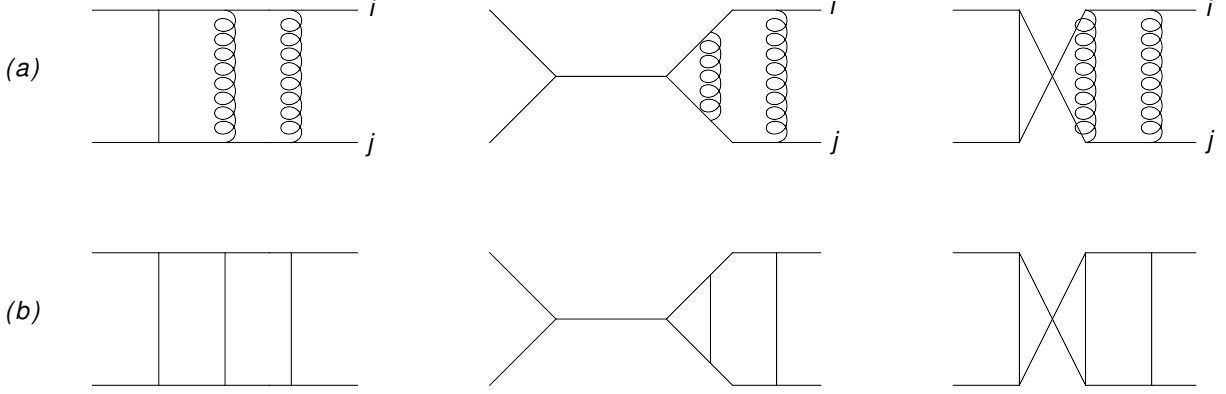


Figure 7: (a) Soft graphs corresponding to eq. (42); (b) full QCD graphs in the corresponding BCJ relation.

The second class of soft BCJ relations at two loop order is more complicated, and consists of sets of three graphs in which the hard part of the interaction (in this case a single tree-level topology) is the same in each term, but the soft graphs are different. All possible examples are shown in figure 8, and the full (non-soft) BCJ triples can be obtained by replacing the soft gluon lines with solid lines, as in figure 7(a) and (b). Note that in figure 8(c), the Y graph is related to itself, analogously to the case of the triangles considered in section 3.1. For BCJ duality to be satisfied, there must exist soft numerators satisfying

$$\begin{aligned}\hat{n}_{x,ijk}^{(Y_3)} + \hat{n}_{x,kji}^{(||_3)} - \hat{n}_{x,ijk}^{(||_3)} &= 0, \\ -\hat{n}_{x,ij}^{(||)} + \hat{n}_{x,ij}^{(X)} + \hat{n}_{x,ij}^{(Y)} &= 0, \\ \hat{n}_{x,ij}^{(O)} &= 0,\end{aligned}\tag{47}$$

corresponding to each of the sets of graphs in figure 8, and where $\hat{n}_{x,i\dots k}^{(Z)}$ is the soft numerator associated with a diagram where soft graph Z dresses tree-level topology x . As is clear from the eikonal integrals of eq. (39), the numerators of the graphs in the Feynman gauge do not automatically satisfy these relations. The numerators are given by

$$\begin{aligned}\tilde{n}_{x,ijk}^{(||_3)} &= p_i \cdot p_j p_j \cdot p_k n_x, \\ \tilde{n}_{x,ijk}^{(Y_3)} &= V_{\mu\nu\rho}(k, l) p_i^\mu p_j^\nu p_k^\rho n_x \\ \tilde{n}_{x,ij}^{(Y)} &= V_{\mu\nu\rho}(k, l) p_i^\mu p_j^\nu p_j^\rho n_x \\ \tilde{n}_{x,ij}^{(||)} &= (p_i \cdot p_j)^2 n_x, \\ \tilde{n}_{x,ij}^{(X)} &= (p_i \cdot p_j)^2 n_x, \\ \tilde{n}_{x,ij}^{(O)} &= V_{\mu\nu\rho}(k, l) V_{\sigma\rho}(k, l) p_i^\mu p_j^\sigma n_x,\end{aligned}\tag{48}$$

where we have used a tilde to denote the fact that these soft numerators do not respect BCJ duality. In order to find those that do, one must effect a generalised gauge transformation $\tilde{n} \rightarrow \hat{n}$. The general form of such a transformation is given in eq. (6), and consists of redefining each numerator

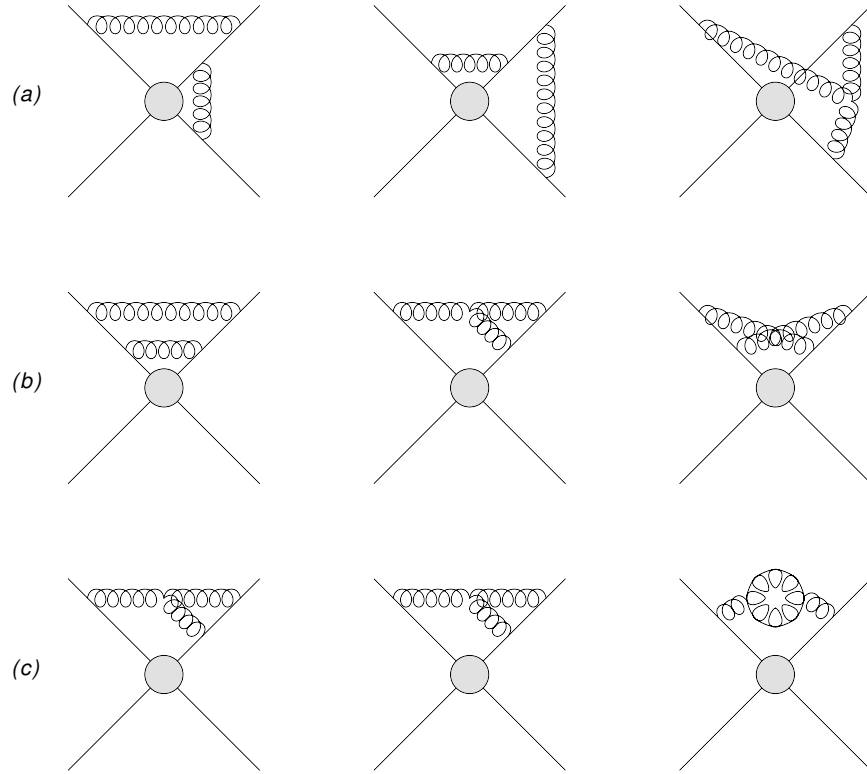


Figure 8: Sets of three graphs, each corresponding to the soft limit of a BCJ relation, such that the hard interaction is the same in each term.

through superpositions of denominator factors. If, however, all we care about is verifying the infrared singularities of gravity via the double copy, *we do not have to solve the BCJ relations explicitly*. To see this, note the numerator factors depend on soft momenta as

$$\tilde{n}_{x,ijk}^{(||_3)}, \tilde{n}_{x,ij}^{(||)}, \tilde{n}_{x,ij}^{(X)} \sim \mathcal{O}(K^0), \quad \tilde{n}_{x,ijk}^{(Y_3)}, \tilde{n}_{x,ij}^{(Y)} \sim \mathcal{O}(K), \quad \tilde{n}_{x,ij}^{(O)} \sim \mathcal{O}(K^2), \quad (49)$$

where $K \equiv (k, l)$ represents a soft momentum scale. Furthermore, all relevant denominator factors scale as at least $\mathcal{O}(K)$. Thus, from eq. (6) one must have

$$\begin{aligned} \hat{n}_{x,ijk}^{(||_3)} &= \tilde{n}_{x,ijk}^{(||_3)} + \mathcal{O}(K), \\ \hat{n}_{x,ij}^{(||)} &= \tilde{n}_{x,ij}^{(||)} + \mathcal{O}(K), \\ \hat{n}_{x,ij}^{(X)} &= \tilde{n}_{x,ij}^{(X)} + \mathcal{O}(K), \end{aligned} \quad (50)$$

and

$$\hat{n}_{x,ijk}^{(Y_3)}, \hat{n}_{x,ij}^{(Y)}, \hat{n}_{x,ij}^{(O)} \sim \mathcal{O}(K). \quad (51)$$

We may summarise this more generally as follows. The BCJ dual numerators for soft graphs involving multiple single gluon emissions between pairs of external lines are the same as those in the Feynman gauge, up to corrections subleading in soft momentum. These corrections will not lead to additional IR singularities, so can be ignored in the soft limit. This is consistent with the first two BCJ relations in eq. (47), which due to the subleading nature of $\hat{n}_{x,ijk}^{(Y_3)}$ and $\hat{n}_{x,ij}^{(Y)}$ amount to

$$\hat{n}_{x,ijk}^{(||_3)} = \hat{n}_{x,kji}^{(||_3)}, \quad \hat{n}_{x,ij}^{(||)} = \hat{n}_{x,ij}^{(X)}, \quad (52)$$

as is indeed observed already in eq. (48).

The BCJ dual numerators for graphs involving three gluon vertices off the eikonal lines are changed with respect to the Feynman gauge numerators, and are first order in soft momentum. Upon taking the double copy to gravity, these numerators will be squared, whereas the denominators are unchanged. Hence, by power counting, such graphs will not contribute infrared singularities in gravity. In other words, the very BCJ relations that one has to invest effort in solving are irrelevant for the double copy! This also tells us that there are infrared singularities in QCD that vanish when one takes the double copy to gravity. This “information loss” will, as we will see, be a feature at higher loop orders.

The above remarks imply that we can take the double copy of eq. (41) by keeping only the graphs with multiple dipole emissions. Their numerators will be given by eq. (50), where we can safely neglect the corrections which are subleading in soft momenta. Performing the double copy procedure gives a 2-loop gravity amplitude ⁹

$$\mathcal{M}_{\text{IR}}^{(2)} = -i \left(\frac{\kappa}{2} \right)^4 \sum_{x \in \{s, t, u\}} \left[\sum_{\langle ijkl \rangle} \mathcal{I}_{ijkl}^{(||_4), \text{grav.}} + \sum_{\langle ijk \rangle} \mathcal{I}_{ijk}^{(||_3), \text{grav.}} + \sum_{\langle ij \rangle} \left(\mathcal{I}_{ij}^{(||), \text{grav.}} + \mathcal{I}_{ij}^{(X), \text{grav.}} \right) \right] \frac{n_x n_x}{x}, \quad (53)$$

⁹For the overall phase factor, see the footnote on p. 12.

where

$$\begin{aligned}
\mathcal{I}_{ijkl}^{(||4),\text{grav.}} &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{4(p_i \cdot p_j)^2 (p_k \cdot p_l)^2}{k^2 l^2 p_i \cdot k p_j \cdot k p_k \cdot l p_l \cdot l}, \\
\mathcal{I}_{ijk}^{(||3),\text{grav.}} &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{4(p_i \cdot p_j)^2 (p_j \cdot p_k)^2}{k^2 l^2 p_i \cdot k p_j \cdot k p_j \cdot (k+l) p_k \cdot l}, \\
\mathcal{I}_{ij}^{(||),\text{grav.}} &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot k p_i \cdot (k+l) p_j \cdot k p_j \cdot (k+l)}, \\
\mathcal{I}_{ij}^{(X),\text{grav.}} &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot k p_i \cdot (k+l) p_j \cdot l p_j \cdot (k+l)}.
\end{aligned} \tag{54}$$

This indeed agrees with an explicit calculation using the known gravitational eikonal factors, obtained using the Feynman rules of eqs. (17) and (18). We have thus shown that, at two loop order, the infrared singularities of GR are consistent with those of QCD via the double copy procedure.

We can write eq. (54) in a more recognisable form as follows. Firstly, one has

$$\mathcal{I}_{ijkl}^{(||4),\text{grav.}} = \mathcal{I}_{ij}^{\text{grav.}} \mathcal{I}_{kl}^{\text{grav.}}, \tag{55}$$

where the right-hand side contains the product of two gravitational one-loop eikonal factors from eq. (33). By collecting terms in eq. (54), we can write the entire right-hand side in terms of one-loop integrals. In particular, one has

$$\begin{aligned}
\mathcal{I}_{ijk}^{(II_3),\text{grav.}} + \mathcal{I}_{kji}^{(II_3),\text{grav.}} &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{4(p_i \cdot p_j)^2 (p_j \cdot p_k)^2}{k^2 l^2 p_i \cdot k p_k \cdot l p_j \cdot (k+l)} \left[\frac{1}{p_j \cdot k} + \frac{1}{p_j \cdot l} \right] \\
&= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{4(p_i \cdot p_j)^2 (p_j \cdot p_k)^2}{k^2 l^2 p_i \cdot k p_k \cdot l p_j \cdot k p_j \cdot l},
\end{aligned} \tag{56}$$

where in the second line we have used the *eikonal identity*

$$\frac{1}{p_j \cdot (k+l)} \left[\frac{1}{p_j \cdot k} + \frac{1}{p_j \cdot l} \right] = \frac{1}{p_j \cdot k p_j \cdot l}. \tag{57}$$

One thus has

$$\mathcal{I}_{ijk}^{(II_3),\text{grav.}} + \mathcal{I}_{kji}^{(II_3),\text{grav.}} = \mathcal{I}_{ij}^{\text{grav.}} \mathcal{I}_{jk}^{\text{grav.}}. \tag{58}$$

Also, one has

$$\begin{aligned}
\mathcal{I}_{ij}^{(||),\text{grav.}} + \mathcal{I}_{ij}^{(X),\text{grav.}} &= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot k p_i \cdot (k+l) p_j \cdot (k+l)} \left[\frac{1}{p_j \cdot k} + \frac{1}{p_j \cdot l} \right] \\
&= \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot k p_i \cdot (k+l) p_j \cdot k p_j \cdot l} \\
&= \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot (k+l) p_j \cdot k p_j \cdot l} \left[\frac{1}{p_i \cdot k} + \frac{1}{p_i \cdot l} \right] \\
&= \frac{1}{2} \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D l}{(2\pi)^D} \frac{4(p_i \cdot p_j)^4}{k^2 l^2 p_i \cdot k p_i \cdot l p_j \cdot k p_j \cdot l}
\end{aligned}$$

$$= \frac{1}{2} \left(\mathcal{I}_{ij}^{\text{grav.}} \right)^2, \quad (59)$$

where we have again used the eikonal identity of eq. (57), as well as relabelling $k \leftrightarrow l$. Putting things together, we may rewrite eq. (53) using the results of eqs. (55, 58, 59) to give

$$\mathcal{M}_{\text{IR}}^{(2)} = -i \left(\frac{\kappa}{2} \right)^4 \sum_{x \in \{s, t, u\}} \left[\sum_{\langle ijkl \rangle} \mathcal{I}_{ij}^{\text{grav.}} \mathcal{I}_{kl}^{\text{grav.}} + \sum_{(ijk)} \mathcal{I}_{ij}^{\text{grav.}} \mathcal{I}_{jk}^{\text{grav.}} + \frac{1}{2} \sum_{\langle ij \rangle} \left(\mathcal{I}_{ij}^{\text{grav.}} \right)^2 \right] \frac{n_x n_x}{x}, \quad (60)$$

where the notation (ijk) denotes summing over all triples, such that the ordering of i and k is unimportant. The total soft prefactor (contents of the square bracket in eq. (60)) is easily checked to be the second order term in the expansion of

$$\exp \left[i \left(\frac{\kappa}{2} \right)^2 \sum_{\langle ij \rangle} \mathcal{I}_{ij}^{\text{grav.}} \right], \quad (61)$$

as expected from the known exponentiation and one-loop exactness of gravitational infrared divergences [49, 50, 52]. This completes our analysis of BCJ duality and the double copy in the soft limit at two-loop order. In the next section, we generalise our remarks to all loop orders.

5 General remarks

In the previous two sections, we have seen that the structure of infrared singularities in QCD matches on to those of GR after applying the double copy procedure. At one loop, we were able to apply the double copy due to the fact that the Feynman gauge numerators for soft graphs automatically satisfied the appropriate BCJ relations in the soft limit. At two loops, BCJ relations could be separated into two classes: (a) those involving graphs whose soft topology was the same, but whose underlying hard tree-level topology was different; (b) those involving graphs sharing the same hard interaction, but having different soft topologies. BCJ relations of class (a) were still automatically satisfied, due to the fact that the underlying tree-level numerators satisfy BCJ duality. Relations of class (b) were more complicated. Numerators for graphs with no three gluon vertices off the external lines (which we refer to as *dipole-like graphs* in the following) could be taken to be the same as the Feynman gauge results up to subleading corrections in soft momenta, which are irrelevant from the point of view of infrared singularities. Numerators of graphs involving three-gluon vertices off the external lines did not automatically satisfy BCJ relations, and thus would have to be modified by generalised gauge transformations in order to write down a BCJ-dual representation of a QCD amplitude in the soft limit. However, these graphs are at least linear in soft momenta, and thus vanish upon taking the double copy to gravity. This allowed us to verify that the IR singularities of gravity are correctly reproduced by double copying the QCD results at two-loop order, without having to explicitly solve the BCJ relations.

The aim of this section is to argue that this argument generalises to all loop orders. This is possible because we have already seen all of the necessary ingredients at two loop order. Firstly, the fact that the infrared limit BCJ relations fall into the two classes given above is generally true, independent of the loop order¹⁰. Then BCJ relations of class (a), and involving a given soft topology (Z),

¹⁰At one loop, as we have seen, only class (a) occurs.

take the generic form of eq. (46), with numerators given by eq. (45). As at two loop order, these relations are satisfied by virtue of the fact that the tree-level numerators satisfy the BCJ relation, and are multiplied by a common factor.

BCJ relations of class (b) are again more complicated, and can be split into two further subclasses: (i) those involving two dipole-like graphs and a graph containing at least one three gluon vertex off the external lines; (ii) those involving three graphs with at least one three gluon vertex off the external lines. By the same power counting arguments that were used in the previous section, the numerators of dipole-like graphs are the same as their Feynman gauge counterparts up to corrections which are subleading in soft momenta (i.e. which do not contribute infrared singularities). Two dipole graphs which enter the same BCJ relation are related by a permutation of two gluon emissions on an external line (e.g. figure 8(a))¹¹. Thus, BCJ relations of subclass (i) set the numerators of such graphs to be equal, which is indeed satisfied in the Feynman gauge, which associates with a single soft gluon emission between lines i and j a contribution $2(p_i \cdot p_j)$, independently of any other gluon emissions. BCJ relations of subclass (ii) are not satisfied by the Feynman gauge numerators for the relevant graphs. However, these numerators are all at least $\mathcal{O}(K)$ (where K is an arbitrary soft momentum), and remain so after performing a generalised gauge transformation in line with eq. (6). Thus, these graphs do not give infrared singularities after performing the double copy. One thus does not need to solve the BCJ relations explicitly in order to generate the IR divergences of the gravity amplitude. From three-loop order in the Feynman gauge, one must also consider graphs involving the four-gluon vertex off the eikonal lines. This also gives rise to numerators which involve non-zero powers of soft momenta, after rewriting such graphs in terms of cubic topologies, according to the usual BCJ procedure.

The above remarks allow us to generalise eq. (60) to an arbitrary loop order, as¹²

$$\mathcal{M}_{\text{IR}}^{(n)} = \left(\frac{\kappa}{2}\right)^{2n} \sum_{x \in \{s, t, u\}} \left[\sum_{m=2}^{2n} \sum_{\langle i_1 \dots i_m \rangle} \tilde{\mathcal{I}}_{i_1 \dots i_m}^{\text{grav.}} \right] \frac{n_x n_x}{x}. \quad (62)$$

In this formula, $\tilde{\mathcal{I}}_{i_1 \dots i_m}^{\text{grav.}}$ is the eikonal integral factor due to the sum of all dipole emissions that link lines $i_1, i_2 \dots i_m$. The second sum in the square brackets in eq. (62) is then over all multipoles $\langle i_1 \dots i_m \rangle$ that are linked by such dipole emissions. The first sum is then over all possible numbers m of external lines. As in the two-loop analysis of the previous section, one may collect terms in eq. (62) into products of one-loop eikonal integrals, via multiple applications of the (higher-order) eikonal identity, and thus rewrite eq. (62) as [49]

$$\mathcal{M}_{\text{IR}}^{(n)} = \left(\frac{\kappa}{2}\right)^{2n} \sum_{x \in \{s, t, u\}} \left[\frac{1}{n!} \left(\sum_{\langle ij \rangle} i \int \frac{d^D k}{(2\pi)^D} \frac{2p_i \cdot p_j}{k^2 p_i \cdot k p_j \cdot k} \right)^n \right] \frac{n_x n_x}{x}. \quad (63)$$

The tree-level amplitude dressed by soft gravitons to all orders, and obtained via the double copy, is then given by

$$\mathcal{M}_{\text{IR}} = \sum_{n=0}^{\infty} \mathcal{M}_{\text{IR}}^{(n)} = \exp \left[\sum_{\langle ij \rangle} i \left(\frac{\kappa}{2}\right)^2 \int \frac{d^D k}{(2\pi)^D} \frac{2p_i \cdot p_j}{k^2 p_i \cdot k p_j \cdot k} \right] \frac{n_x n_x}{x}, \quad (64)$$

¹¹In the language of [29, 58, 59], such diagrams are in the same multiaprtion web.

¹²Here we have absorbed factors of i into the eikonal integrals, rather than show these explicitly as in eq. (1).

in agreement with the known all-order structure of IR divergences in GR [49]. Note that, once we had reached eq. (62), the final result was guaranteed: what mattered was that the double copy procedure correctly reproduces the fact that only dipole-like graphs appear in gravity. The sum over all such graphs automatically leads to the exponentiation of the one-loop corrections ¹³.

This completes our argument that the all-order structure of IR divergences in gravity is consistent with IR singularities in QCD, via the application of the double copy procedure. This can therefore be taken as all-order evidence for the double copy conjecture, albeit in a particular limit. Some further comments are in order. Firstly, the general argument at $\mathcal{O}(\alpha_S^n)$ exhibits a property already remarked upon at two loops, namely that many singularities cancel upon taking the double copy to gravity. This means that the gravity side of the correspondence cannot be used to constrain singularities on the gauge theory side. For this reason, the question posed earlier in the paper regarding whether the QCD dipole formula has a gravitational origin appears to have a negative answer: the very singularities which could occur as corrections to the dipole formula *vanish* when we move to gravity.

Secondly, we have here focused on the case of pure Yang-Mills theory and General Relativity. However, the argument of this paper can also in principle be applied in supersymmetric gauge theories / supergravity. Here we remark that reference [16] obtained amplitudes in $\mathcal{N} \geq 4$ supergravity using the double copy procedure applied to gauge theory amplitudes in $\mathcal{N} = 4$ Super-Yang-Mills theory coupled with $0 \leq \mathcal{N} \leq 4$ Super-Yang-Mills theory. One check on this calculation was the demonstration that infrared singularities, after the double copy, were consistent with exponentiation on the gravity side up to two loop order. The results of this paper generalise this to all loop orders, and non-supersymmetric theories. In supersymmetric theories, it may be possible to extend our arguments beyond the pure soft limit, as non-trivial information can in principle be obtained from infrared singularities [66, 67].

6 Conclusion

In this paper, we have examined the soft limits of QCD (strictly speaking, pure gluodynamics) and GR, and showed that infrared singular contributions to amplitudes in both theories match up with each other upon using the double copy procedure of [2, 3]. The structure of IR divergences in QCD scattering amplitudes is still subject to some uncertainty, and our current state of knowledge can be expressed in terms of the dipole formula of [31–34], plus possible corrections [35–39]. By contrast, the structure of IR singularities is known exactly in GR [49–52], where they exponentiate in terms of one-loop graphs only.

Being able to take the double copy relies on the gauge theory amplitudes displaying manifest BCJ duality [5]. At one-loop order, we saw that amplitudes in the soft limit, as calculated in the Feynman gauge, automatically satisfied BCJ duality. This was not the case at two-loop order. By power counting, however, this turns out to be irrelevant for graphs consisting of multiple dipole emissions. These are the only graphs that survive upon taking the double copy, so that one does

¹³This is the same argument that occurs in the exponentiation of one-loop soft corrections in QED [20, 49]. In that theory, however, one-loop exactness is broken due to the presence of fermion bubbles.

not need to solve the BCJ relations explicitly. We could thus use the double copy procedure to “predict” the structure of IR singularities in GR, finding exact agreement with the known results. Our arguments imply that many singularities vanish upon taking the double copy. If this were not the case, one could have used singularities in gravity to constrain possible corrections to the QCD dipole formula, or even to provide an underlying gravitational explanation for this. Nevertheless, the arguments presented in this paper constitute all-loop level evidence for the validity of the double copy conjecture, which may be more significant in supersymmetric contexts.

Note that we have only considered the case where the hard interaction consists of tree-level scattering. In principle, this could contain higher loop orders. To show that BCJ duality is satisfied by the full amplitude then requires BCJ-dual numerators for the hard interaction, which is not possible to all orders without a full proof of BCJ duality in QCD.

There are a number of further questions that can be addressed. It may be possible, for example, to reinterpret our results using the manifestly BCJ-dual effective Lagrangian of [3]. It would also be interesting to examine the full implications of the present analysis in supersymmetric contexts. Finally, a thorough investigation of the role of BCJ duality in non-supersymmetric gauge theory away from the soft limit would potentially provide new insights into QCD and / or gravity. An intermediate step in this regard might be to extend the present analysis to next-to-eikonal order, using the technology of [51, 62, 63].

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A Numerator reflection properties under particle interchange

In this appendix, we show that the numerators obtained for the 1-loop triangle graphs in the soft limit obey the BCJ reflection property of eq. (5), namely that if the colour factor changes by a sign under interchanging two (external or internal) legs, the numerator also does so. It will be sufficient to consider a particular example, for which we choose the graph labelled (T1) in figure 3.

In the soft limit, this graph may be redrawn as shown in figure 9, where we have assigned a particular momentum flow. The labels on the legs can be translated into physical momenta as follows:

$$3 \equiv k, \quad 1 = 2 \equiv p_1, \quad 4 = 5 \equiv p_2, \quad (65)$$

using the fact that the soft gluon momentum $k \rightarrow 0$. The numerator for this graph in the soft limit is given by eq. (30) as

$$\hat{n}_{T1}(1, 2, 3, 4, 5) = (p_1 \cdot p_2) n_s(2, 5), \quad (66)$$

where in the tree-level numerator for the s -channel graph we note explicitly that this has the momenta 2 and 5 as its incoming legs.

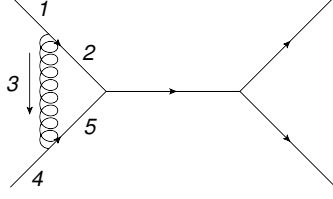


Figure 9: The soft limit of the triangle diagram (T1) in figure 3.

The colour factor of this graph picks up a minus sign under the interchanges $2 \leftrightarrow 5$, $2 \leftrightarrow 3$ and $3 \leftrightarrow 5$. Making the first interchange and imposing momentum conservation gives

$$3 \equiv k, \quad 1 = 5 \equiv p_2, \quad 2 = 4 \equiv p_1. \quad (67)$$

The numerator factor itself maps to

$$\hat{n}_{T1}(1, 2, 3, 4, 5) \xrightarrow{2 \leftrightarrow 5} (p_1 \cdot p_2) n_s(5, 2). \quad (68)$$

We may now use the fact that the tree-level numerator is BCJ dual, and thus must satisfy

$$n_s(2, 5) = -n_s(5, 2). \quad (69)$$

Substituting this into eq. (68) then yields

$$\hat{n}_{T1}(1, 2, 3, 4, 5) \xrightarrow{2 \leftrightarrow 5} -\hat{n}_{T1}(1, 2, 3, 4, 5)$$

as required.

Next, we consider the interchange of legs $2 \leftrightarrow 3$, which modifies the graph shown in figure 9 to give that of figure 10(a), where we have included relevant physical momenta, after imposing momentum conservation. The resulting graph is not infrared singular as $k \rightarrow 0$. Indeed, the k dependence of the original graph

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (p_1 - k)^2 (p_2 + k)^2}$$

have been shifted to

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k + p_1)^2 (k - p_2)^2}.$$

The infrared singular region is found by shifting $k \rightarrow k + p_1$, after which one obtains the graph shown in figure 10(b), which has k dependence

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{k^2 (k + p_1)^2 (k - p_2)^2}.$$

This corresponds to the original graph, with $p_1 \leftrightarrow p_2$. Put another way, in order to make figure 10(b) have the same momentum flow as figure 9, we must flip the two incoming lines. Thus, one has

$$\hat{n}_{T1}(1, 3, 2, 4, 5) = p_1 \cdot p_2 n_s(5, 2), \quad (70)$$

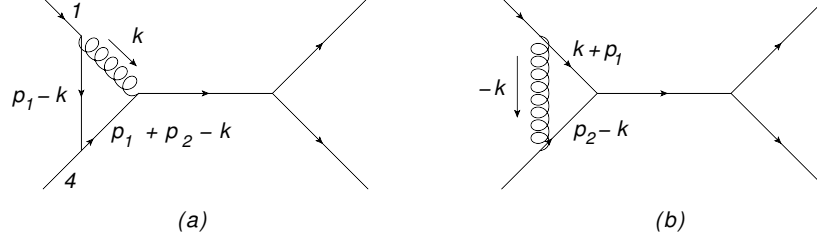


Figure 10: (a) Diagram obtained by interchanging legs $2 \leftrightarrow 3$ in figure 9; (b) Diagram obtained by shifting $k \rightarrow k + p_1$.

where we have reinstated the notation used in figure 9. Again using eq. (69), we find

$$\hat{n}_{T1}(1, 3, 2, 4, 5) = -\hat{n}_{T1}(1, 2, 3, 4, 5) \quad (71)$$

as required. The case of $3 \leftrightarrow 5$ is directly analogous.

Here we have considered a triangle built upon the s -channel tree-level graph. A similar analysis can be used for the t - and u -channel cases, as well as triangular subgraphs at higher loop orders.

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